A NOTE ON BALANCED ORTHOGONAL CHEMICAL BALANCE WEIGHING DESIGNS

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SUMMARY

A series of chemical balance weighing designs were proposed in this note through balanced orthogonal designs which are balanced and orthogonal. The efficiencies of these designs are compared with the alternative designs given by Dey [2] and Meena [4].

Introduction

Weighing designs were first proposed by Hotelling [3] with an idea to increase the precision of weighing light objects on a chemical balance. Let there be v objects to be weighed on a chemical balance and b weighings ($v \leqslant b$) are made to estimate the weights of the v objects.

Let the observational equations be

$$Y = x\beta + e$$

where Y is the $(b \times 1)$ observation vector, $\underline{x} = (x_{ij})$, $i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$, is the design matrix, $\bar{\beta}$ is the vector of weights to be estimated and $\underline{e}$ is the random vector of errors distributed with mean zero and constant variance σ^2. In the design matrix, x_{ij} takes values -1, 1, 0 according as the ith object is in the left pan, right pan or not included in the jth weighing. If $\mathbf{x'x}$ is of full rank, the estimates were given by $\boldsymbol{B} = (X'X)^{-1} X'Y$ with covariance matrix $(X'X)\sigma^2$.

Designs for chemical balance weighing problems were proposed by several authors. Banerjee [1] used the incidence matrix of the BIB design in constructing weighing designs. He suggested to repeat the design to have some degrees of freedom for error also. Dey [2] and Meena [4] proposed some alternatives to repeated designs. In the present note a class of weighing designs is proposed. These designs are derived from balance orthogonal designs due to Rao [6] and Mukhopadhyay [5].

Designs Through BOD $(s^2 + s + 1, s^2, s^2 - s)$

Rao [6] defined balanced orthogonal design (BOD) as a matrix of order $v \times b$ with $(\pm 1, 0)$ satisfying the following conditions:

(i) inner product of any two rows of it is zero;
(ii) when -1's of it are changed to 1's, the resulting matrix becomes incidence matrix.

He discussed several methods of constructing these designs and in particular he showed that the BOD with parameters $v = s^2 + s + 1 = b$; $r = s^2 = k$; $\lambda = s^2 - s$ exists whenever s is an odd prime power. This series of BOD's gives us chemical balance weighing designs. If a BOD is taken as such the resulting design will be saturated leaving no degrees of freedom for error and therefore, the whole design i.e. the BOD may be repeated to obtain the new design D as shown below:

$$D = \frac{(B)}{(-B)}$$

In this design every object is weighed an equal number of times in the left and right pans. The variance-co-variance matrix of the design, given by $\sigma^2 Iv/2s^2$ is diagonal indicating that the weights can be estimated orthogonally.

Meena [4] proposed a chemical balance weighing design D_0 with $(2v - 1)$ weighing operations and compared the efficiency with the designs suggested by Dey [2]. The design D_0 is obtained as follows.

Assuming, without loss of generality, that the r $(=k)$ elements of the first block B_1 of the chosen BIB design represent the first r of the v objects to be weighed. Deleting the block B_1, we shall be left with $(v - 1)$ blocks in each of which there will be exactly λ elements from among the first r objects of the BIB design. Obtain $(v - 1) \times v$ matrix N_0 from the incidence matrix N^* of the chosen BIB design by performing the following operations: (i) delete the first row of N^*, (ii) replace 0 by -1 and (iii) in each row of the resulting matrix replace by 0's the λ unities corresponding to the λ elements the row has from among the first r objects. Then the design D_0 is given by

$$D_0 = \left[\frac{N_1}{N_0} \right]$$

where N_1 is given by replacing 0 by -1 in N^*. It can be verified that

$$D_0' D_0 = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

where $A = 5(r - \lambda)I_r + (2v - 6r + 5\lambda)J_{r.r}$

$B = (2v - 7r + 6\lambda)J_{r \cdot (v-r)}$

$C = 8\,(r - \lambda)I_{v-r} + \{2v - 1 - 8(r - \lambda)\}J_{(v-r)\,(v.r)}$

The efficiency of D proposed here can be compared with D_0 for the series (S_2: $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$, s being a prime power) under the three criteria, viz.,

(i) Minimum variance
(ii) A-optimality
(iii) D-optimality

from the values given in the following table. It can be seen that D is superior to D_0 under criteria (i) and (ii). However, D_0 is superior, in general, under criteria (iii).

Designs Through BOD $(s + 1, s, s - 1)$

Seberry (1978) showed that the BOD exists with parameters ($s^2 + s + 1, s^2, s^2 - s$) when s is an even prime number also. Subsequently Mukhopadhyay [5] generalised these results and showed that the series of BOD's $(s^{p+3} - 1)/(s - 1)$, s^{p+2}, $s^{p+2} - s^{p+1}$ exists for all $p \geqslant -1$ when s is an odd prime or prime power. The series exists for $p = 0$, when s is an even prime power. When $p = 0$ the BOD parameters are $s^2 + s + 1$, s^2, $s^2 - s$) through which the construction of weighing design had been discussed in the previous section.

Now let us consider the series when $p = -1$. In this case the parameters of BOD are $(s + 1)$, s, $s - 1$) where s is an odd prime or prime power. As earlier let the BOD be repeated to get the design D, then the variance-covariance matrix, the trace of $(X'X)^{-1}$ and the det of $(X'X)$ are respectively

given by $\frac{1}{2s}\ Iv$; $\frac{s+1}{2s}$ and $(2s)^{s+1}$. With these values it can be seen

that these designs are more efficient compared to D_0 (Meena, 1981) and D_1 to D_5 (Dey, [2]). Further, designs for s objects can also be derived from this series by omitting a column and they can be seen to be more efficient

TABLE

Design	Variance factor for first r objects	Variance factor for the last (v − r) objects.	Trace of $(X'X)^{-1}$	Det $(X'X)$
D_0	$\frac{9s^3-14s^2-27s+48}{5(9s^4-15s^3-11s^2+16s+8)}$	$\frac{9s^4-15s^3-20s^3+47s+3}{8s(9s^4-15s^3-11s^2+16s+8)}$	$\frac{45s^5-3s^4-140s^3-99s^2+183s+384}{40(s^2+s+1)(9s^4-15s^3-11s^2+16s+8)}$	$5^s 8^{s^2-1} s^{s^2+s}(9s^4-15s^3-11s^2+16s+8)$
D	$\frac{1}{2s^2}$	$\frac{1}{2s^2}$	$\frac{s^2+s+1}{2s^2}$	$(2s^2)^{s^2+s+1}$

compared to the available designs at the cost of two more additional weighings.

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